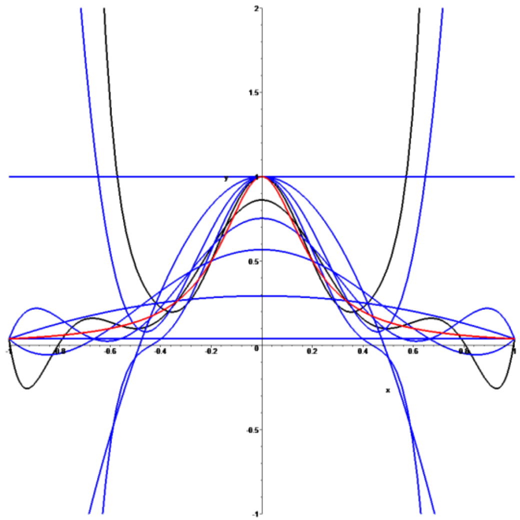
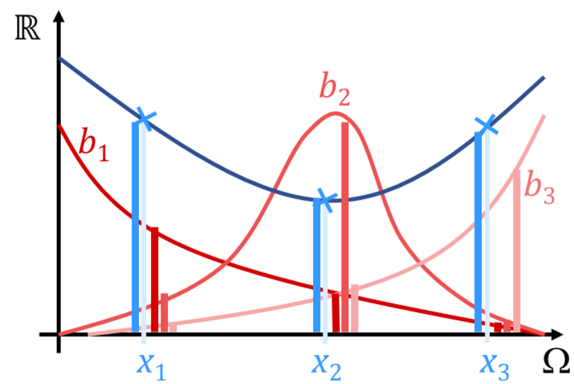
Modelling 1

SUMMER TERM 2020

Institut für Informatik

JOHANNES GUTENBERG UNIVERSITÄT MAINZ





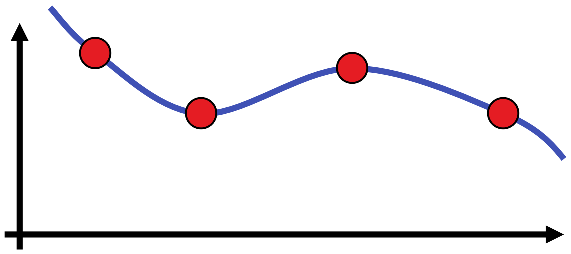
LECTURE 13

Interpolation

Michael Wand · Institut für Informatik · Michael.Wand@uni-mainz.de

Interpolation

Interpolation Problem

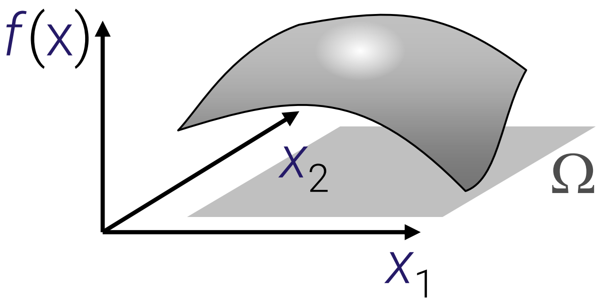


Interpolation with a “Linear Ansatz”

- Given a set of points
- Choose basis functions
  - Properties of basis determine result
- Find a linear combination that interpolates

General Formulation

Settings:



- Domain $\Omega \subseteq \mathbb{R}^d$. Mapping:

$$f: \Omega \rightarrow \mathbb{R}$$

- Basis:

$$B = \{b_1, \ldots, b_n\}, b_i: \Omega \rightarrow \mathbb{R}$$

- f as linear combination of basis functions:

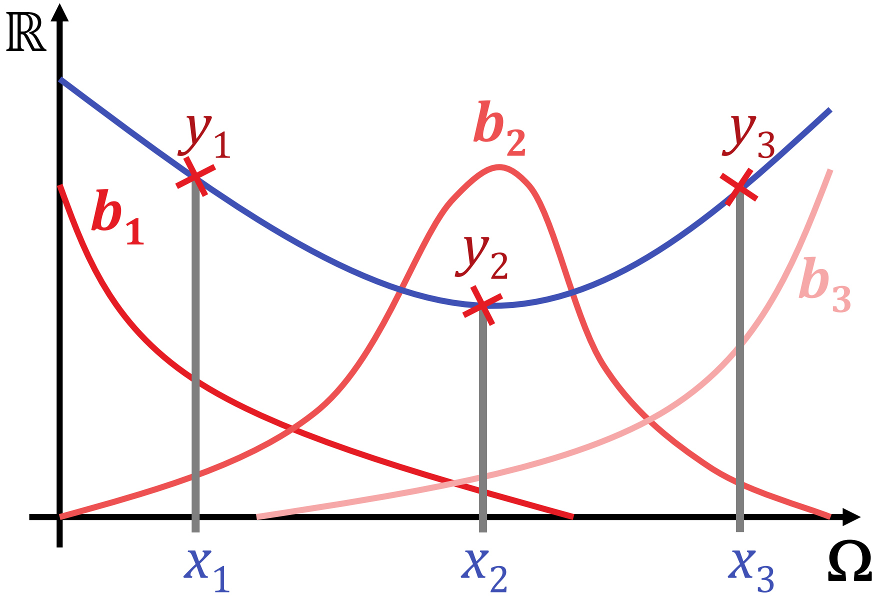
$$f_\lambda = \sum_{i=1}^{n} \lambda_i b_i$$

- Function values: $\{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)\} \subset \mathbb{R}^d \times \mathbb{R}$

- Find λ such that: $\forall i = 1, \ldots, n: f_\lambda(\mathbf{x}_i) = y_i$

1D Example

$$f: \mathbb{R} \rightarrow \mathbb{R}$$



Solving the Interpolation Problem

Solution: linear system of equations

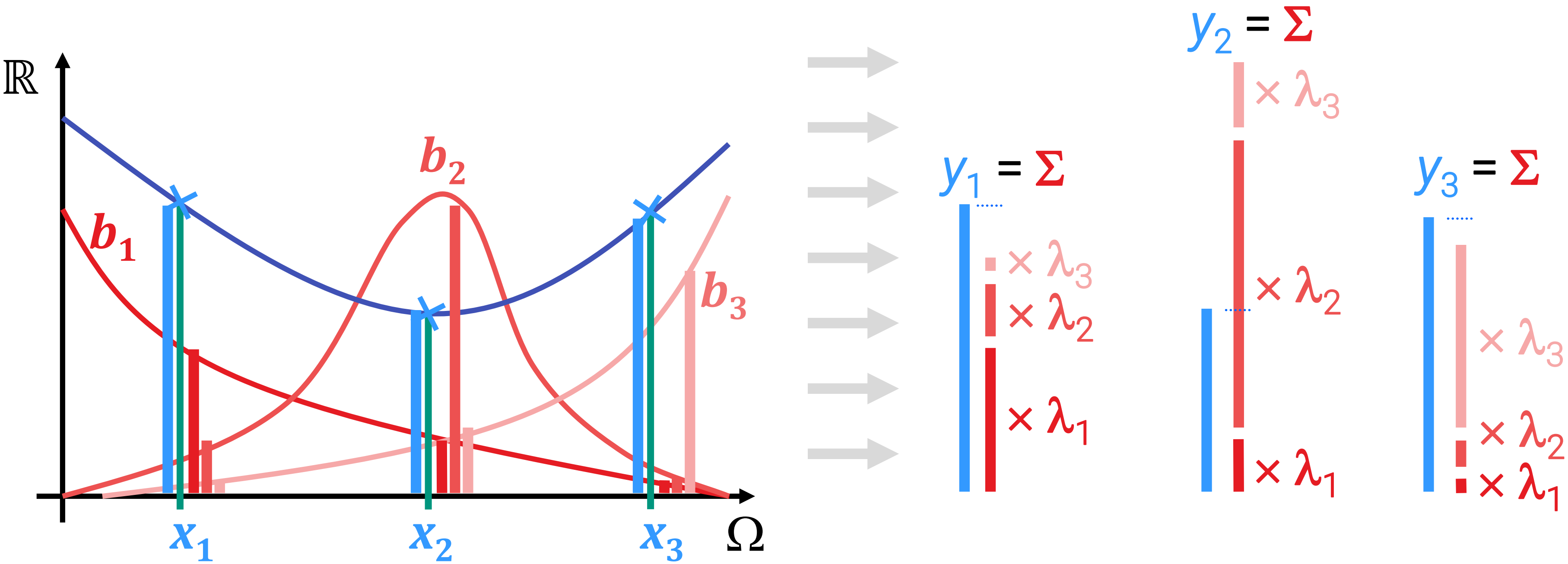
- Evaluate basis functions at points $\mathbf{x}_i$:

$$\forall i = 1, \dots, n: \sum_{i=1}^{d} \lambda_i b_i(\mathbf{x}_i) = y_i$$

- Matrix form:

$$\begin{pmatrix} b_1(\mathbf{x}_1) & \cdots & b_n(\mathbf{x}_1) \\ \vdots & & \vdots \\ b_1(\mathbf{x}_n) & \cdots & b_n(\mathbf{x}_n) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

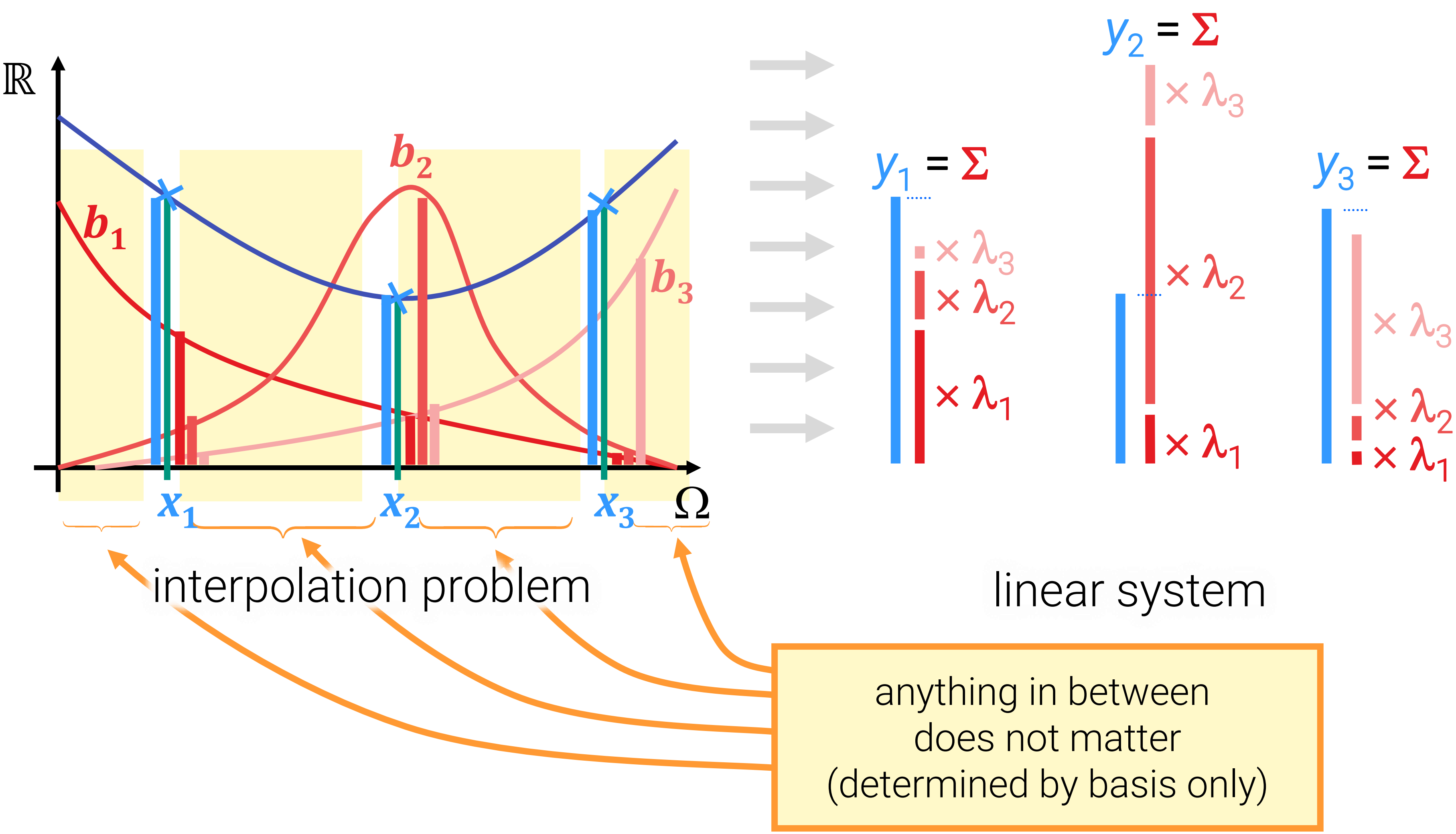
Illustration



interpolation problem

linear system

Illustration



Example

Example: Polynomial Interpolation

- Monomial basis $B = \{1, x, x^2, x^3, \ldots, x^{n-1}\}$
- Linear system to solve:

$$\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

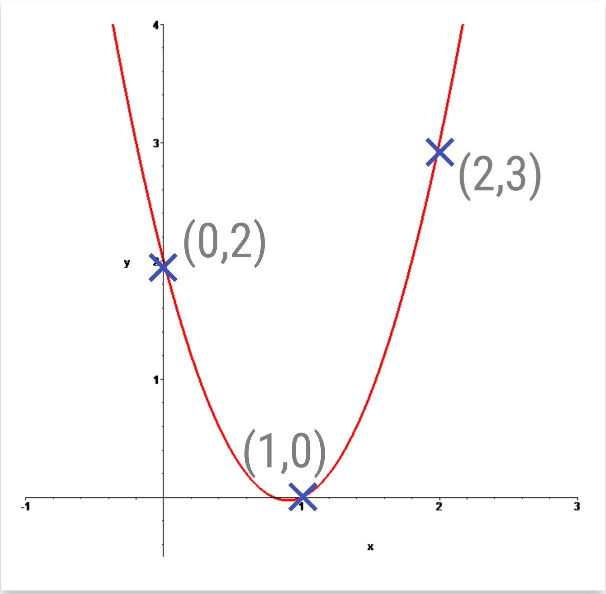
"Vandermonde Matrix"

Example with Numbers

Example with numbers

- Quadratic monomial basis $B = \{1, x, x^2\}$
- Function values: $\{(0,2), (1,0), (2,3)\}$
- Linear system to solve:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

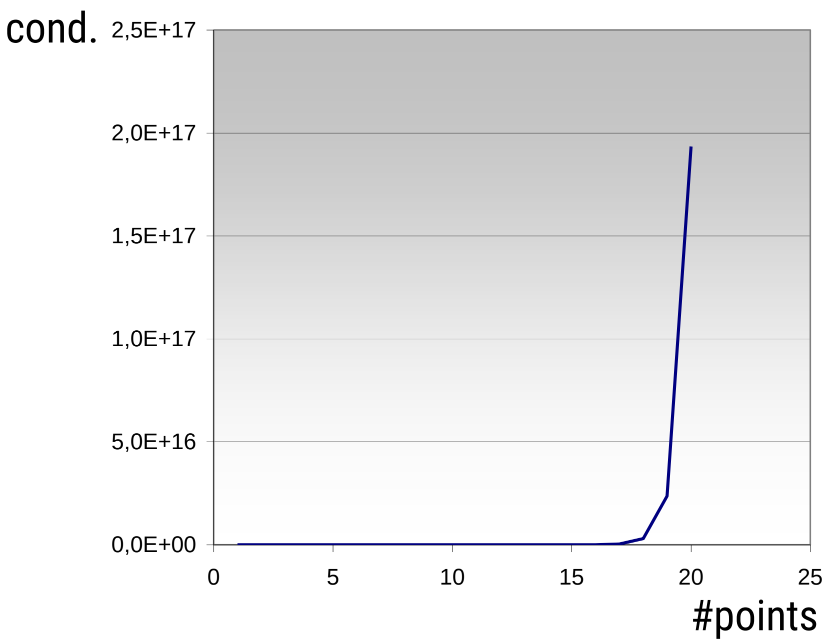


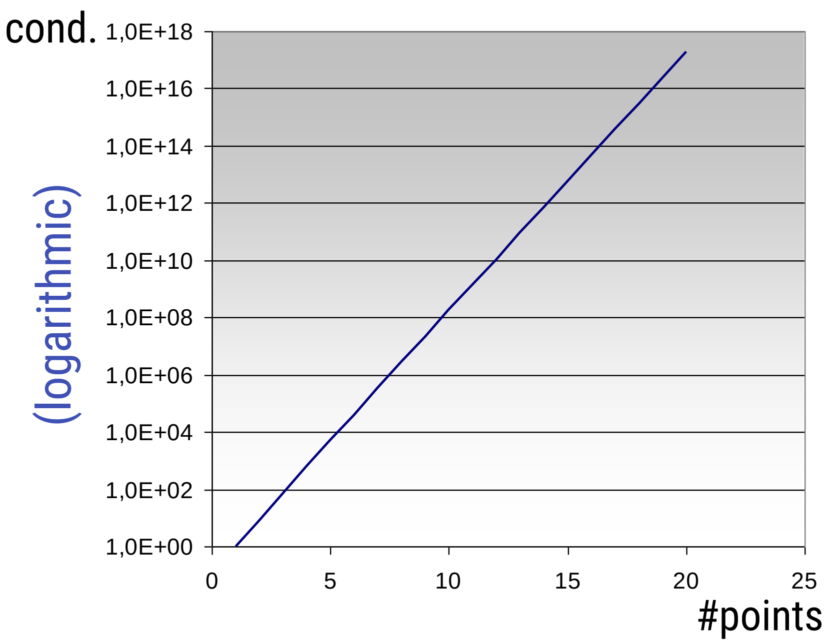
- Result: $\lambda_1 = 2, \lambda_2 = -\frac{9}{2}, \lambda_3 = \frac{5}{2}$

Condition Number...

Monomial interpolation ill conditioned

- Vandermonde matrix, equidistant x_i
- Condition number grows exponentially with n

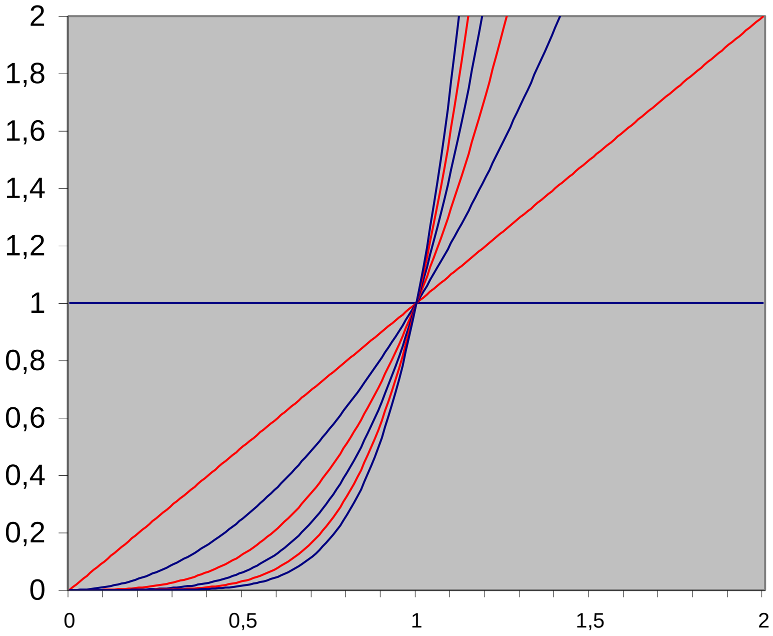




Why is that?

Monomial Basis:

- Increasingly indistinguishable
- Difference in growth rate

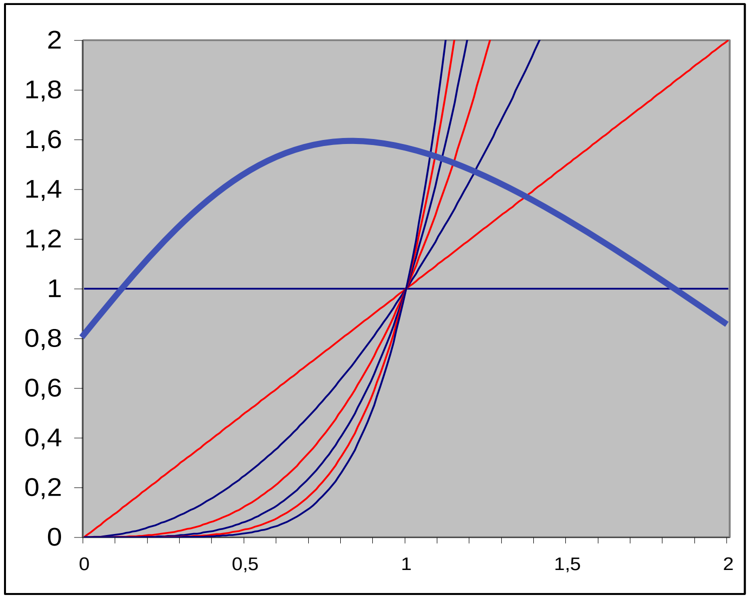


Monomial basis

Cancellation

Monomials:

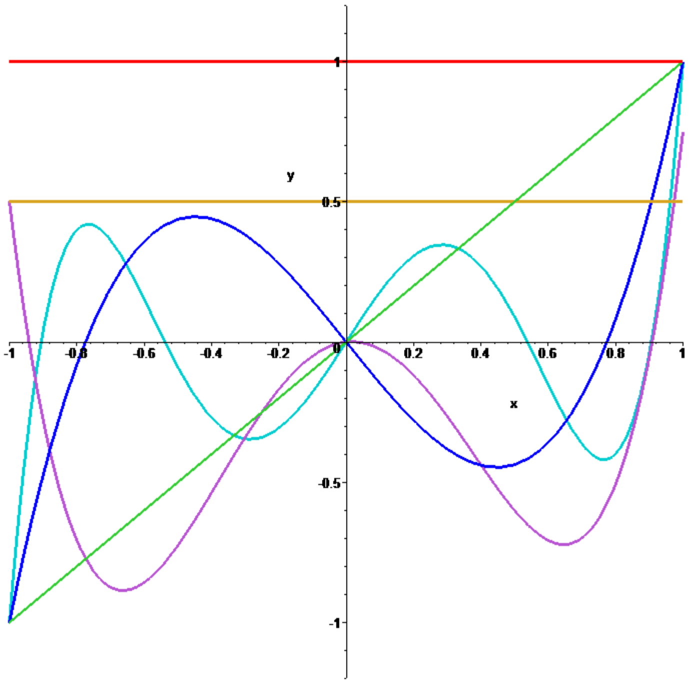
- From left to right in x-direction...
- First 1 dominates
- Then x grows faster
- Then x^2 grows faster
- Then x^3 grows faster
- ...



The Cure...

This problem can be fixed:

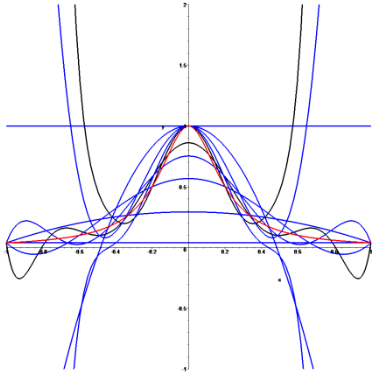
- Orthogonal polynomial basis
- E.g.: Gram-Schmidt orthogonalization
- Legendre polynomials – orthonormal on [-1..1]



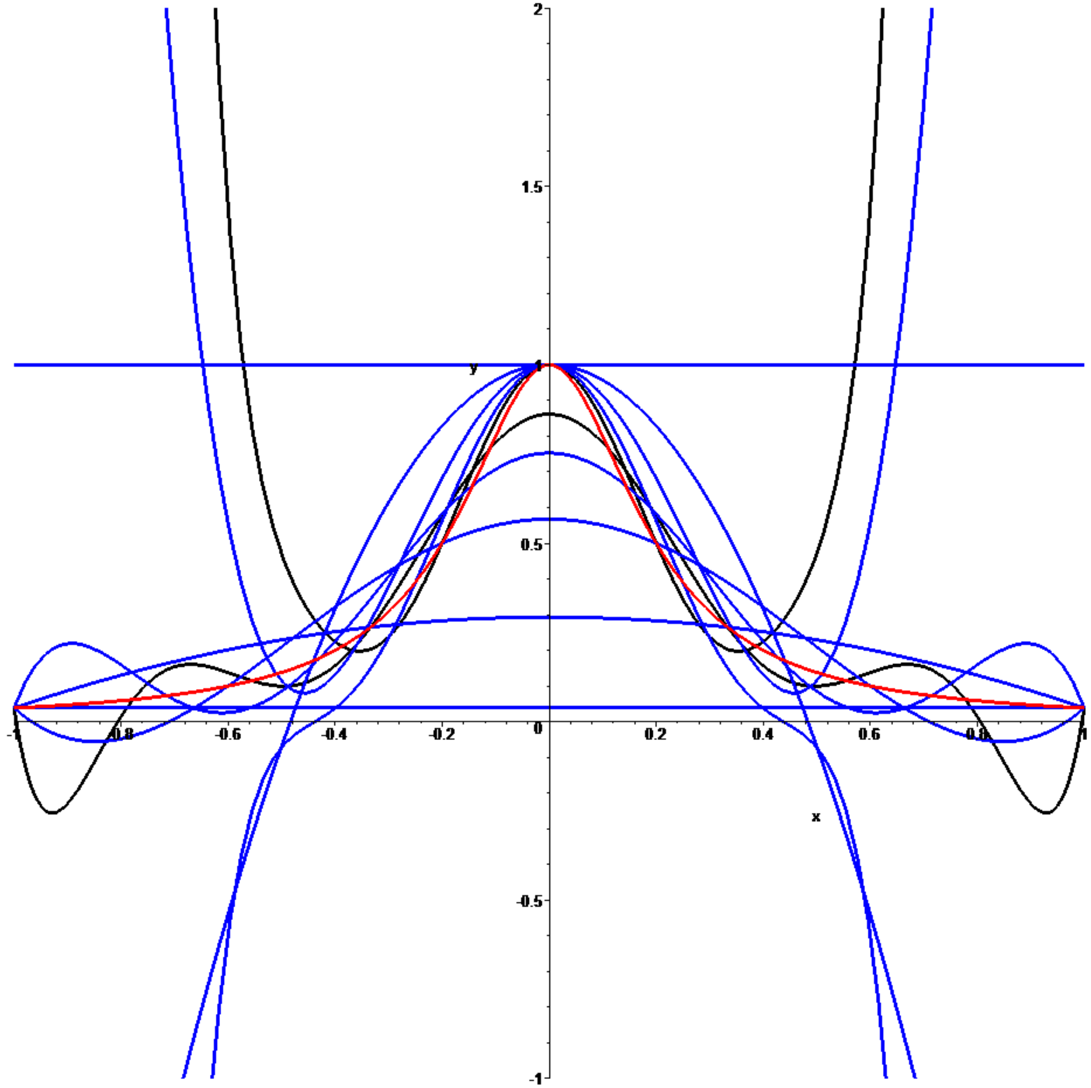
However...

This does not fix all problems:

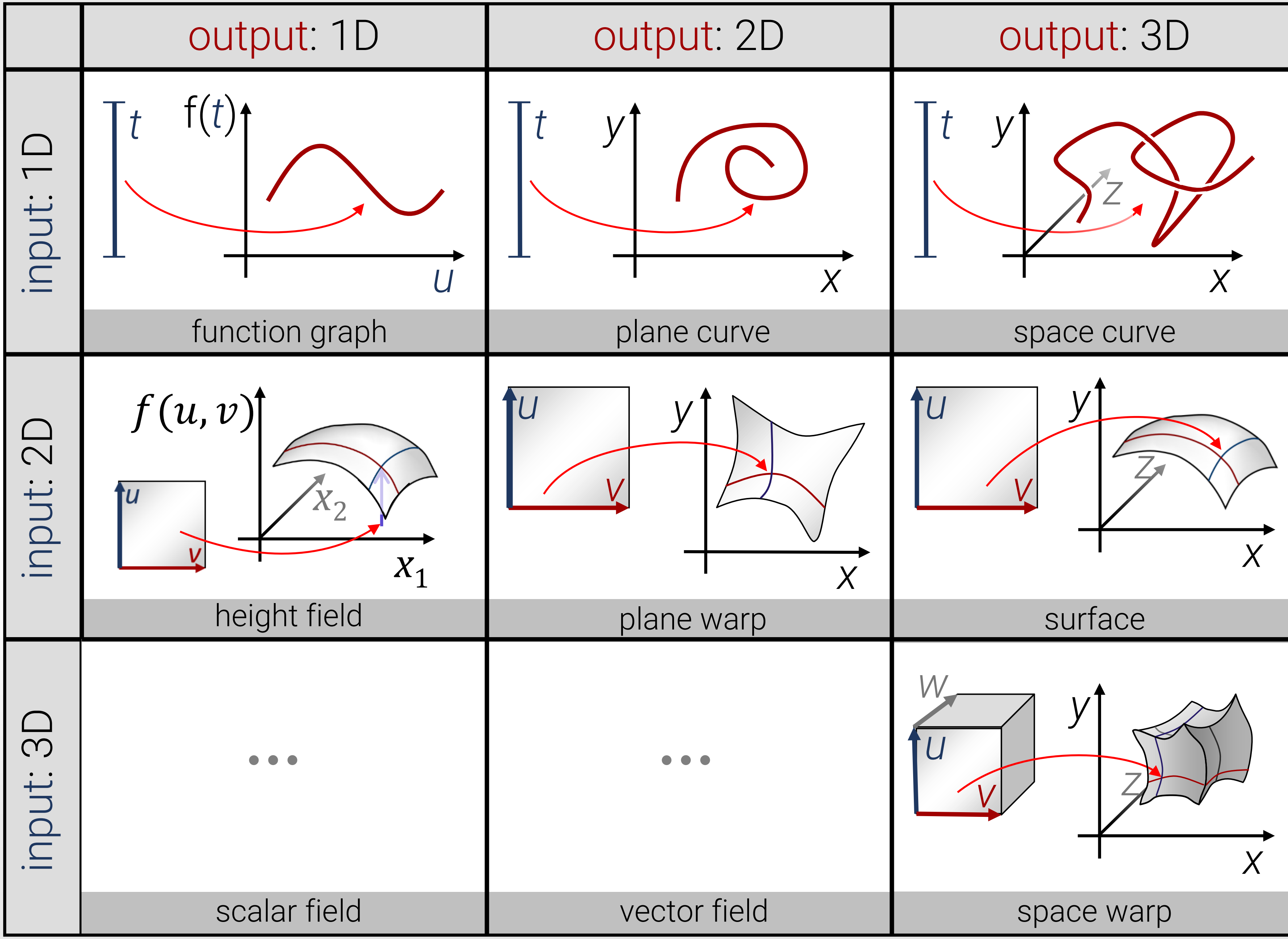
- Polynomial interpolation is instable
  - "Runge's phenomenon"
    - Uniformly spaced control points
    - Oscillating behavior
- Weierstraß approximation theorem:
  - Smooth functions (C^0): uniform convergence with sequence of polynomals
  - Need to chosen polynomials very carefully
  - Not very useful in practice



Runge’s Phenomenon



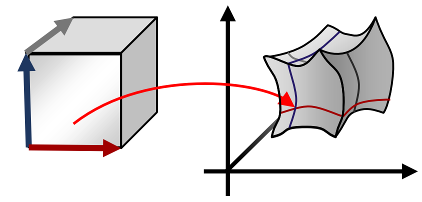
Generalizations



output: 1D
output: 2D
output: 3D
input: 1D
input: 2D
input: 3D
t
f(t)
u
y
x
z
function graph
plane curve
space curve
f (u, v)
u
v
x2
x1
height field
plane warp
surface
w
scalar field
vector field
space warp

Solving the Interpolation Problem

Multi-Dimensional output



- For every target coordinate k:

$$\forall j = 1, \dots, n: \sum_{i=1}^{d} \lambda_i^{(k)} b_i^{(k)} (\mathbf{x}_j) = y_j^{(k)}$$

- Solve a separate interpolation problem for each target dimension

Differential / Integral Constraints

Taking linear operators of target function

- Consider linear operator L

$$L f(\mathbf{x}_j) = \mathbf{y}_j \quad \rightarrow \quad \sum_{i=1}^{n} \lambda_i (L b_i)(\mathbf{x}_j) = \mathbf{y}_j$$

- Still a linear system
  - Just apply L to basis functions
- Differential & integral linear operators

$$\frac{d}{dx}, \qquad \frac{\partial}{\partial_x}, \qquad \Delta = \frac{\partial^2}{\partial_x^2} + \frac{\partial^2}{\partial_y^2}, \qquad f \rightarrow \int_a^b f(x) w(x) dx, \ldots$$